Th1 Let $\emptyset \neq D \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$. Then TFSA: 1

(i) x_0 is a cluster pt (~~or~~ non-isolated point) w.r.t D:

$\forall \delta > 0 \ \exists x \in D$ s.t. $0 < |x - x_0| < \delta$.

(ii) $\exists$ a seq (x_n) in $D \setminus \{x_0\}$ s.t. $\lim_n x_n = x_0$

(iii) $\operatorname{dist}(x_0, D\setminus\{x_0\}) = 0$ where

$$\operatorname{dist}(x_0, D\setminus\{x_0\}) := \inf\{\|x_0 - x\| : x \in D \setminus \{x_0\}\}$$

Henceforth, let $f: D \to \mathbb{R}$ $(i=1,2)$ and let $x_0 \in \mathbb{R}$ be a cluster pt w.r.t D.

Th2 (Boundedness Th). Let $\emptyset \neq D \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$ a cluster-point w.r.t D. Let $f: D \to \mathbb{R}$, and suppose $l := \lim_{x \to x_0} f(x)$ exists in $\mathbb{R}$. Then $\exists \ \delta > 0$ and $M > 0$ such that

$$(\#) \quad |f(x)| \leq M \quad \forall x \in V_\delta(x_0) \cap D$$

Proof. Let
$$M := \begin{cases} |l| + 1 & \text{if } x_0 \notin D \\ \max\{|l|+1, |f(x_0)|\} & \text{if } x_0 \in D \end{cases}$$

Will show that (#) holds for some $\delta > 0$. Indeed, with $\varepsilon := 1$, $\exists \ \delta > 0$ such that

$$|f(x) - l| < 1 \quad \forall x \in (V_\delta(x_0) \setminus \{x_0\}) \cap D$$

By Δ-ineq it follows that

$$|f(x)| < |l| + 1 \leq M \quad \forall x \in (V_\delta(x_0) \setminus \{x_0\}) \cap D$$

Since, if $x = x_0 \in D$, one also has $|f(x)| = |f(x_0)| \leq M$ by definition of M, we see that (#) holds.

Th3 (Order-preserving) Let $f: D \to \mathbb{R}$, $x_0 \in \mathbb{R}$ a cluster pt w.r.t. D. Suppose

$$\alpha < \lim_{x \to x_0} f(x) < \beta$$

Then $\exists\, \delta > 0$ s.t.

$$\alpha < f(x) < \beta \quad \forall\, x \in \left(V_\delta(x_0) \setminus \{x_0\}\right) \cap D.$$

Proof. Let $\varepsilon = \min\{\beta - l, l - \alpha\}$ where $l := \lim_{x \to x_0} f(x)$.
Then $\varepsilon > 0$ and $\exists\, \delta > 0$ s.t.

$$l - \varepsilon < f(x) < l + \varepsilon \quad \forall\, x \in \left(V_\delta(x_0) \setminus \{x_0\}\right) \cap D$$

Noting $l + \varepsilon \leq l + (\beta - l) = \beta$ and

$$\alpha = l - (l - \alpha) \leq l - \varepsilon$$

it follows that

$$\alpha < f(x) < \beta \quad \forall\, x \in \left(V_\delta(x_0) \setminus \{x_0\}\right) \cap D.$$

Remark. Allow the possibility that $\alpha = -\infty$ or $\beta = +\infty$.

Cor. Suppose $f(x) \geq \beta$ $\forall\, x \in \left(V_{\delta_1}(x_0) \setminus \{x_0\}\right) \cap D$ where $\delta_1 > 0$, and suppose that $l := \lim_{x \to x_0} f(x)$. Then $l \geq \beta$. (similar result for inequality $f(x) \leq \alpha$ ...)

Pf. Suppose not: $l < \beta$ (so $l - 1 < l = \lim_{x \to x_0} f(x) < \beta$)

Then $\exists\, \delta_2 > 0$ s.t.

$$l-1 < f(x) < \beta \quad \forall\, x \in (V_{\delta_2}(x_0) \setminus \{x_0\}) \cap D$$

Pick $\bar{x} \in (V_{\delta_1 \wedge \delta_2}(x_0) \setminus \{x_0\}) \cap D$ (as x_0 is cluster w.r.t. D). Then $f(\bar{x}) < \beta$, contradicting the given assumption.

Th 4. Suppose $l = \lim_{x \to x_0} f(x) \neq 0$. Then $\exists\, \delta > 0$ s.t.

$$(*) \quad \frac{|l|}{2} < |f(x)| < \frac{3|l|}{2} \quad \forall\, x \in (V_\delta(x_0) \setminus \{x_0\}) \cap D.$$

Proof. Let $\varepsilon := \frac{|l|}{2}$ (>0). Then $\exists\, \delta > 0$ s.t.

$$|f(x) - l| < \frac{|l|}{2} \qquad \forall\, x \in (V_\delta(x_0) \setminus \{x_0\}) \cap D$$

Since $|f(x)| - |l| \leq |f(x) - l|$ and $|l| - |f(x)| \leq |f(x) - l|$, $(*)$ follows. ▭

Remark. You can also show that $|l| = \lim_{x \to x_0} |f(x)|$ and apply Th 3 to $\alpha := |l|/2$ and $\beta := 3|l|/2$.

Th 5 (Computation Rules) e.g.

$$\lim_{x \to x_0} \frac{f_1(x)}{f_2(x)} = \frac{\lim_{x \to x_0} f_1(x)}{\lim_{x \to x_0} f_2(x)}$$

if $f_2(x) \neq 0 \neq l_2 := \lim_{x \to x_0} f_2(x)$ $(\in \mathbb{R})$ $\forall\, x \in D$ and $l_1 := \lim_{x \to x_0} f_1(x)$ exists in $\mathbb{R}$ (algebraic $+, -, \times, \div$ and lattice operations)

Proof. Let $\delta_3 > 0$ be s.t.

$$\frac{|l_2|}{2} < |f_2(x)| < \frac{3|l_2|}{2} \quad \forall x \in (V_{\delta_3}(x_0) \setminus \{x_0\}) \cap D$$

Let $\varepsilon > 0$ be given. Take $\varepsilon_1, \varepsilon_2 > 0$ s.t.

$$\varepsilon_1, \varepsilon_2 \leqslant \frac{\varepsilon \cdot |l_2|^2}{4M} \quad \text{where } M := \max\{|l_1|, |l_2|\} (>0)$$

For these $\varepsilon_1, \varepsilon_2$ $\exists$ $\delta_1, \delta_2 > 0$ s.t., for each $i = 1, 2$,

$$|f_i(x) - l_i| < \varepsilon_i \quad \forall x \in (V_{\delta_i}(x_0) \setminus \{x_0\}) \cap D$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ (>0). Then $\forall x \in (V_\delta(x_0) \setminus \{x_0\}) \cap D$ one has $|f_i(x) - l_i| < \varepsilon_i$ and $|f_2(x)|^{-1} \leqslant \frac{1}{|l_2|/2}$

$$\left|\frac{f_1(x)}{f_2(x)} - \frac{l_1}{l_2}\right| = \frac{|f_1(x) l_2 - l_1 l_2 + l_1 l_2 - f_2(x) l_1|}{|f_2(x)|\,|l_2|}$$

$$\leqslant \frac{|l_2|\,|f_1(x) - l_1| + |l_1|\,|f_2(x) - l_2|}{|l_2|^2/2}$$

$$< \frac{2M(\varepsilon_1 + \varepsilon_2)}{|l_2|^2} \leqslant \frac{4M(\varepsilon |l_2|^2 / 4M)}{|l_2|^2} = \varepsilon.$$

This shows that $\displaystyle\lim_{x \to x_0} \frac{f_1(x)}{f_2(x)} = l_1 / l_2$.

Th6. Let $f(x) \geqslant 0$, $l \geqslant 0$ and $\displaystyle\lim_{x \to x_0} f(x) = l$. Then $\displaystyle\lim_{x \to x_0} \sqrt{f(x)} = \sqrt{l}$.

Proof. Separately consider 2 cases:

(i). Case that $l = 0$

(ii) $l > 0$.

The reader is asked to complete the proof.

Th 7. (Order-preserving & Squeeze Principle).

(i) Suppose $\exists \delta > 0$ s.t. $f_1(x) \le f_2(x)$ $\forall x \in (V_\delta(x_0) \setminus \{x_0\}) \cap D$ and $l_i = \lim_{x \to x_0} f_i(x) \in \mathbb{R}$. Then $l_1 \le l_2$.

(by Cor. of Th 3).

(ii) Suppose $\exists \delta > 0$ s.t. $f_1(x) \le f(x) \le f_2(x)$ $\forall x \in (V_\delta(x_0) \setminus \{x_0\}) \cap D$ and that $l_1 = l_2$ $(= l \text{ say})$ where $l_i = \lim_{x \to x_0} f_i(x)$, $i = 1, 2$.

Then $\lim_{x \to x_0} f(x) = l$.

Proof of (ii): Let $\varepsilon > 0$. Then $\exists \delta_i > 0$ $(i = 1, 2)$ s.t.

$$l - \varepsilon < f_i(x) < l + \varepsilon \quad \forall x \in (V_{\delta_i}(x_0) \setminus \{x_0\}) \cap D$$

Let $\delta' = \min\{\delta_1, \delta_2, \delta\}$. Since $f_1(x) \le f(x) \le f_2(x)$ $\forall x \in (V_\delta(x_0) \setminus \{x_0\}) \cap D$ and $V_{\delta_i}(x_0), V_\delta(x_0) \supseteq V_{\delta'}(x_0)$ it follows that

$$l - \varepsilon < f_1(x) \le f(x) \le f_2(x) < l + \varepsilon \quad \forall x \in (V_{\delta'}(x_0) \setminus \{x_0\}) \cap D$$

This shows that $\lim_{x \to x_0} f(x) = l$.

About notations

① $l := \lim_{x \to x_0} f(x)$

[I don't like: $\lim_{x \to x_0} f$]
but Bartle uses at times

② $l := \lim_{\substack{x \to x_0 \\ x \in A}} f(x)$

with the understanding that

(i) x_0 is a cluster w.r.t. ① the domain of f
② $\{x : x \in A \ \& \ x \in \text{domain of } f\}$

e.g. $\lim_{\substack{x \to x_0 \\ x > x_0}} f(x)$ (right-sided limit)

(ii) the limit does exist (and hence unique, due to the cluster property.